

Equivalence Theorem of Uncertainty Relations

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Abstract

We present an equivalence theorem to unify the two classes of uncertainty relations, i.e., the variance based ones and the entropic forms, which shows that the entropy of an operator in a quantum system can be built from the variances of a set of commutative operators. That means an uncertainty relation in the language of entropy may be mapped onto a variance based one, and vice versa. Employing the equivalence theorem, new entropic uncertainty relations stronger than existing ones in the literature are obtained for qubit system, and variance based uncertainty relations for spin systems are reached from the corresponding entropic uncertainty relations.

1 Introduction

The renowned uncertainty principle is one of the distinctive features of quantum mechanics, which was introduced by Heisenberg in the description of microscopic quantum behavior [1], somewhat similar to the concept of complementary raised by Bohr [2]. The

uncertainty relation (UR) is a mathematical expression for uncertainty principle, referring to the repulsive nature of incompatible operators and hence imposing a strong restriction on the outcomes of any joint measurement on those operators. Since the UR has profound influence on various aspects of quantum information, e.g. quantum nonlocality [3–5], entanglement [6], and quantum cryptography [7], the study on it has never stopped.

It is well-known that the most famous uncertainty relation, the Heisenberg-Robertson uncertainty relation [8], bounds the product of the variances of observables A and B through the expectation value of their commutator, i.e.,

$$\Delta A \Delta B \geq |\langle C \rangle| . \quad (1)$$

Here, the state $|\psi\rangle$ is arbitrary, $\Delta X = \sqrt{\langle \psi | X^2 | \psi \rangle - \langle \psi | X | \psi \rangle^2}$ is the square root of the variance of a given operator X , and $[A, B] = 2iC$ is the commutator of operators A and B . Note, the relation (1) is applicable to any pairs of operators, rather than only to conjugate observables. Though later improvement [9, 10] strengthened the uncertainty relation, the state-dependent feature of lower bound remains, which implies the null lower bound triviality [11]. Early attempts in searching the state-independent lower bounds led only to near-optimal results [12]. In a recent work [13], this triviality problem was solved at length, by which the state-independent optimal trade-off relations for the variances of multiple observables are found to be obtainable, at least in principle is, see also [14] for further development of uncertainty relations involving variances of multiple observables.

It was noticed that the variance is inadequate in quantifying uncertainty, i.e. labelling of the non-degenerate eigenvalues of an operator may alter the value of its variance [11]. To overcome this problem, the concept of entropy was employed, and a typical uncertainty relation in entropy tells [15]

$$H(A) + H(B) \geq -2 \ln c_{ab} . \quad (2)$$

Here, $H(A) = -\sum_j p_j \ln p_j$ is the Shannon entropy, with p_j the probability distribution of the eigenbasis $\{|a_j\rangle\}$ of operator A in measuring system, and similarly for $H(B)$. The bound $c_{ab} = \text{Max}_{j,k} |\langle a_j | b_k \rangle|$ is the maximum overlap of eigenbases of operators A and B and independent of the quantum state. To construct an optimal entropic uncertainty relation, the key point is to find the best lower bound, which usually is a tough question for general observables [16, 17] and so far only the bounds for qubit system with Shannon [18–20] and collision entropies [21] were obtained (see also [22] and references there in). The entropic uncertainty relation may in principle apply to multiple observables, of which in fact the state-independent lower bounds have been investigated for mutually unbiased bases [23, 24], see Ref.[25] for a recent review.

There are actually many types of entropies capable of characterizing the quantum uncertainty [22], e.g., Rényi entropies $H_\alpha(A) = \ln(\sum_j p_j^\alpha)/(1-\alpha)$ with different indices of positive real numbers $\alpha \in \mathbb{R}^+$ (when $\alpha \rightarrow 1$ it is Shannon entropy). As there is no obvious reason why one type of entropy is superior to others in the context of uncertainty relation, a new characterization of uncertainty was introduced: the majorization of the probability distribution [26–28], which is closely related to the entropy. Since both variance and entropy originate from the probability distribution of measurement, one may naturally ask: are these two types of uncertainty relations relevant or equivalent? We find in this work that the answer is definite.

In the following we will present a general scheme on how to build quantitative relations between two prominent uncertainty measures, the variance and the entropy, which indicates that those two classes of uncertainty relations may actually be unified. The scheme can be sketched as follows: first construct a set of commutative operators for a given physical observable, then reconstruct the probability distribution of the measure-

ment outcomes of the physical observable from the variances of operators in the set. Based on the quantitative relation, we get various entropic uncertainty relations for qubit system from variance based uncertainty relations, which give optimal lower and upper bounds for arbitrary entropic measures and for multiple observables beyond mutually unbiased bases. Moreover, new variance based uncertainty relations for high spin systems are also obtained from the entropic uncertainty relations.

2 The equivalence theorem and its applications

In quantum mechanics, a physical system can be generally described by density operator ρ , which is a positive definite Hermitian matrix; and a physical observable is represented by a Hermitian operator and may be expressed through the spectral decomposition $A = \sum_{j=1}^N \lambda_j |j\rangle\langle j|$, where $|j\rangle$ are the eigenvectors of A with the corresponding eigenvalues λ_j . When measuring observable A in a quantum system ρ , only its eigenvalues λ_j are attainable with certain probability in every individual measurement. In ρ ensemble, the probability of measuring λ_j reads $p_j = \langle j|\rho|j\rangle$. This statistical interpretation leads to two uncertainty measures, the variance and entropy, which are mathematically expressed as

$$V(A) = \frac{1}{2} \sum_{j,k=1}^N p_j p_k (\lambda_j - \lambda_k)^2, \quad (3)$$

$$H_\alpha(A) = \frac{1}{1-\alpha} \ln \left(\sum_{j=1}^N p_j^\alpha \right). \quad (4)$$

Here, $V(A)$ signifies variance defined as $V(A) \equiv \Delta A^2 = \text{Tr}[\rho A^2] - \text{Tr}[\rho A]^2$, and $H_\alpha(A)$ represents the Rényi entropy. Notice that subtracting a constant from the operator does not change its variance and entropy, we are hence legitimate to treat the operators in following discussion to be traceless.

2.1 The equivalence theorem

Following we exhibit an equivalence theorem, the main result of this work, which may quantitatively relate different uncertainty measures of discrete systems.

Theorem 1 *For a given physical observable A in N -dimensional representation with eigenbases $|j\rangle$, there exists a set of commutative operators, $\mathcal{A} = \{A_i | A_i = \sum_{j=1}^N \lambda_j^{(i)} |j\rangle\langle j|, A_1 = A\}$, whose variances in quantum state ρ are*

$$\Delta A_i^2 = \sum_{k>j=1}^N p_j p_k g_{jk}^{(i)}, \text{ with } g_{jk}^{(i)} = (\lambda_j^{(i)} - \lambda_k^{(i)})^2, \quad (5)$$

from which the probability distribution $p_j = \langle j | \rho | j \rangle$ could be uniquely determined. The infimum of the cardinality of the set \mathcal{A} lies in $[N-1, N(N-1)/2]$.

Proof: Let $l = (j-1)N + k - (j+1)j/2$, then Eq. (5) may be rewritten as

$$\Delta A_i^2 = \sum_{l=1}^{N(N-1)/2} G_{il} x_l, \quad (6)$$

where $G_{il} = g_{jk}^{(i)}$ and $x_l = p_j p_k$ with $k > j$. The number of linear equations (6) equals to the cardinality of the set \mathcal{A} which we denote as $|\mathcal{A}|$. When $|\mathcal{A}| = N(N-1)/2$, the coefficient matrix G_{il} can be constructed to be invertible by assigning specific values to $\lambda_j^{(i)}$ for $i = 1, 2, \dots, N(N-1)/2$. The solutions of x_l are linear functions of ΔA_i^2 , which in turn yields $N(N-1)/2$ equations for p_j

$$p_j p_k = x_l(\Delta A_1^2, \dots, \Delta A_{N(N-1)/2}^2). \quad (7)$$

Here p_j can be uniquely determined as functions of ΔA_i^2 .

As Eq. (7) is an over determined equation system, we need not to know all the $N(N-1)/2$ variables of x_l to uniquely determine the N variables p_j . That means the set

\mathcal{A} may be even constructed with $|\mathcal{A}| \leq N(N-1)/2$. On the other hand, the number of equations constraining p_j cannot be less than N , otherwise the solution of p_j will not be unique. Considering the additional constraint $\sum_{j=1}^N p_j = 1$, $|\mathcal{A}|$ must be greater than or equal to $N-1$, the dimension of Cartan subalgebra of $SU(N)$ group. In all, the cardinality of the set $|\mathcal{A}|$ lies in $[N-1, N(N-1)/2]$. Q.E.D.

Theorem 1 applies for arbitrary physical observables. When the observable is non-degenerate, i.e., $\forall i \neq j, \lambda_i \neq \lambda_j$, the commutative set \mathcal{A} may be constructed explicitly and the following proposition holds.

Proposition 1 *For non-degenerated observable A in N -dimensional representation with eigenbases $|i\rangle$, the probability distribution $p_i = \langle i|\rho|i\rangle$ in a quantum state ρ may be expressed in terms of covariance functions*

$$p_i^2 = \frac{\Omega_{ij}\Omega_{ik}}{\Omega_{jk}}, \quad (8)$$

where $\Omega_{1k} \equiv \sum_{j=2}^N \text{cov}(\ell_k, \ell_j)$, $2 \leq k \leq N$, $\Omega_{jk} \equiv -\text{cov}(\ell_j, \ell_k)$, $2 \leq j < k \leq N$, with the covariance function $\text{cov}(\ell_i, \ell_j) = \langle \ell_i(A)\ell_j(A) \rangle - \langle \ell_i(A) \rangle \langle \ell_j(A) \rangle$, and the Lagrange basis polynomials $\ell_j(x) = \prod_{\substack{m=1 \\ m \neq j}}^N \frac{x - \lambda_m}{\lambda_j - \lambda_m}$.

Proof: The least degree polynomial function, that assumed to be valued as $f(\lambda_i)$ for N distinct λ_i , is a linear combination of Lagrange basis polynomials $f(x) = \sum_{j=1}^N f(\lambda_j)\ell_j(x)$, where $\ell_j(x) = \prod_{\substack{m=1 \\ m \neq j}}^N \frac{x - \lambda_m}{\lambda_j - \lambda_m}$. The variance of the operator $f(A)$ may be expressed as

$$\Delta f(A)^2 = \sum_{k>j=1}^N p_j p_k [f(\lambda_j) - f(\lambda_k)]^2, \quad (9)$$

according to Eq.(3). By setting $f(\lambda_1) - f(\lambda_k) = \alpha_{k-1}$, we have

$$\Delta f(A)^2 = \sum_{j=2}^N p_1 p_j \alpha_{j-1}^2 + \sum_{k>j=2}^N (\alpha_{j-1} - \alpha_{k-1})^2 p_j p_k. \quad (10)$$

On the other hand, because $\Delta f(A)^2 = \Delta f'(A)^2$ where $f'(A) = f(A) - f(\lambda_1)$, we have

$$\begin{aligned} \Delta f(A)^2 &= \sum_{i=2}^N \alpha_{i-1}^2 \sum_{j=2}^N \text{cov}(\ell_i, \ell_j) - \\ &\quad \sum_{n>m=2}^N (\alpha_{m-1} - \alpha_{n-1})^2 \text{cov}(\ell_m, \ell_n) . \end{aligned} \quad (11)$$

Here $f'(x) = -\sum_{i=2}^N \alpha_{i-1} \ell_i(x)$ is employed, and $\text{cov}(\ell_i, \ell_j) = \langle \ell_i(A) \ell_j(A) \rangle - \langle \ell_i(A) \rangle \langle \ell_j(A) \rangle$.

The equivalence of Eqs. (10) and (11) does not depend on the values of α_i , therefore

$$p_1 p_j = \sum_{k=2}^N \text{cov}(\ell_j, \ell_k) ; p_j p_k = -\text{cov}(\ell_j, \ell_k) , \quad k > j \geq 2 , \quad (12)$$

which leads to Eq. (8). Q.E.D.

Next, we shall illustrate the extraordinary function of the equivalence theorem in bridging the prevailing variance and entropy uncertainty relations through concrete examples of spin systems.

2.2 Uncertainty relations for qubits

Qubit system might be the mostly investigated system in quantum information, which possesses enormous potential in application. In such systems, any physical observable may be represented by a 2×2 traceless Hermitian matrix, and therefore the eigenvalues of an operator may be assigned as $\lambda_2 = -\lambda_1 = \lambda$. According to Proposition 1 the following corollary holds.

Corollary 1 *In a qubit system, there exists the following monotonic functional relations between the entropy and the variance*

$$H_\alpha(A) = f_\alpha(\Delta A^2) = \frac{1}{1-\alpha} \ln(A_+^\alpha + A_-^\alpha) , \quad (13)$$

$$\Delta A^2 = f_\alpha^{-1}[H_\alpha(A)] \equiv g_\alpha(A) , \quad (14)$$

where $A_{\pm} \equiv (1 \pm \sqrt{1 - \Delta A^2})/2$ with the eigenvalues of A being absorbed into its variance $\Delta A^2/\lambda^2 \rightarrow \Delta A^2$, and f_{α}^{-1} is the inverse function of f_{α} .

Proof: For qubit system where $N = 2$, we have $\ell_2(A) = (A - \lambda_1)/(\lambda_2 - \lambda_1)$, and

$$p_1 p_2 = \text{cov}(\ell_2, \ell_2) = \frac{\Delta A^2}{4\lambda^2}. \quad (15)$$

Absorbing λ^2 into ΔA^2 , and considering of $p_1 + p_2 = 1$, then

$$p_1 = \frac{1 + \sqrt{1 - \Delta A^2}}{2}, \quad p_2 = \frac{1 - \sqrt{1 - \Delta A^2}}{2}. \quad (16)$$

Substituting Eq. (16) into the definition of Rényi entropy Eq. (4), we have

$$H_{\alpha}(A) = f_{\alpha}(\Delta A^2) = \frac{1}{1 - \alpha} \ln(A_+^{\alpha} + A_-^{\alpha}), \quad (17)$$

where $A_{\pm} \equiv (1 \pm \sqrt{1 - \Delta A^2})/2$. Eq. (17) is a monotonic function for $\Delta A^2 \in [0, 1]$, and therefore

$$\Delta A^2 = f_{\alpha}^{-1}[H_{\alpha}(A)] \equiv g_{\alpha}(A). \quad (18)$$

Here f_{α}^{-1} is the inverse function of the Rényi entropy with index α . Q.E.D.

Corollary 1 predicts that, an entropic uncertainty relation may be converted into a variance based uncertainty relation straightforwardly. For example, putting Eq. (16) into the entropic uncertainty relation Eq. (2) we get

$$A_+^{A_+} \cdot A_-^{A_-} \cdot B_+^{B_+} \cdot B_-^{B_-} \leq c_{ab}^2, \quad (19)$$

where the quantities $A_{\pm} = (1 \pm \sqrt{1 - \Delta A^2})/2$, $B_{\pm} = (1 \pm \sqrt{1 - \Delta B^2})/2$. Similarly, taking Eq. (16) into the recently found majorized uncertainty relation [26], we have

$$(1 + \sqrt{1 - \Delta A^2})(1 + \sqrt{1 - \Delta B^2}) \leq (1 + c_{ab})^2. \quad (20)$$

Here c_{ab} is defined in Eq. (2).

On the other hand, the variance based uncertainty relation may also be transformed into entropic uncertainty relation. However, the state-dependence of the lower bounds of the variance based uncertainty relations leads to trivial entropy relations, and the non-trivial results only exist for state-independent ones. For example, we have the following entropic uncertainty relation by taking Eq. (14) into the Theorem 1 of Ref. [13]

$$\begin{aligned} [a^2(p^2 - 1) + g_\alpha(A)][b^2(p^2 - 1) + g_\beta(B)] \geq \\ [\sqrt{a^2 - g_\alpha(A)}\sqrt{b^2 - g_\beta(B)} - \kappa p^2]^2. \end{aligned} \quad (21)$$

Here, $a^2 = \text{Tr}[A^2]/2$, $b^2 = \text{Tr}[B^2]/2$, $p^2 = 2\text{Tr}[\rho^2] - 1$, and $\kappa = \text{Tr}[AB]/2$; α and β are independent Rényi indices. Eq. (21) gives both the optimal lower and upper bounds for arbitrary entropic measures, and therefore is superior to all the existing results in the literature [18–22]. To show this more explicitly, we take pure quantum system with operators $A = \vec{\sigma} \cdot \vec{n}_a$, $B = \vec{\sigma} \cdot \vec{n}_b$ and Shannon entropies of $\alpha = \beta = 1$ as an example. In this case, Eq. (21) becomes

$$g_1(A)g_1(B) \geq [\sqrt{1 - g_1(A)}\sqrt{1 - g_1(B)} - \cos \theta_{ab}]^2, \quad (22)$$

where θ_{ab} is the angle between unit vectors \vec{n}_a , \vec{n}_b . Fig. 1 illustrates the allowed regions for the Shannon entropies of operators A and B predicted by Eq. (22) which is derived from the variance based uncertainty relation. These figures are consistent with the recent results obtained by analyzing the parameters of state space of qubit [31].

For observables more than two, the following corollary exists:

Corollary 2 *In a qubit system, for three independent observables $A = \vec{\sigma} \cdot \vec{n}_a$, $B = \vec{\sigma} \cdot \vec{n}_b$, and $C = \vec{\sigma} \cdot \vec{n}_c$, where \vec{n}_a , \vec{n}_b , and \vec{n}_c are not coplane, the entropic uncertainty relation involving $H_\alpha(A)$, $H_\beta(B)$, and $H_\gamma(C)$ where $\alpha, \beta, \gamma \in \mathbb{R}^+$, takes the form of equality.*

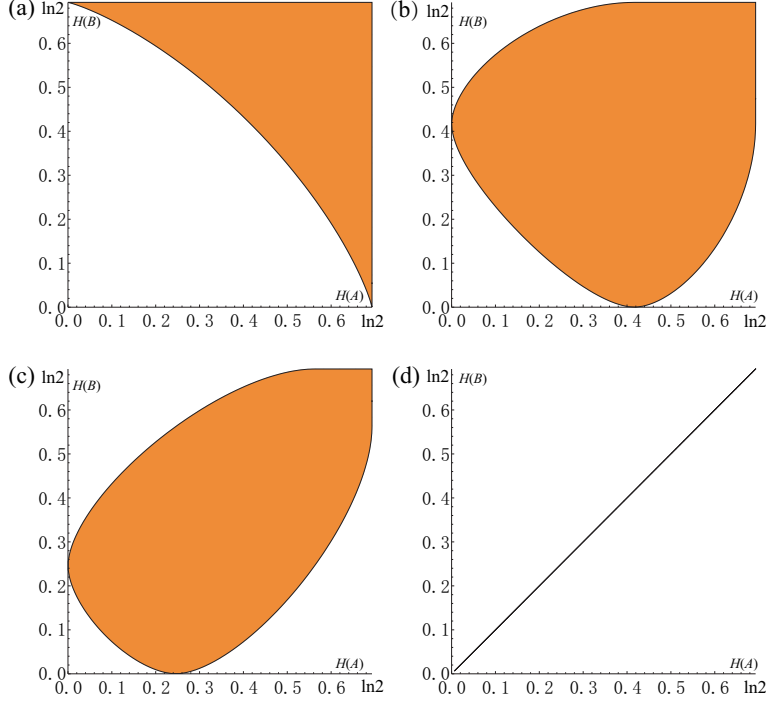


Figure 1. The allowed regions for the Shannon entropies of operators $A = \vec{\sigma} \cdot \vec{n}_a$ and $B = \vec{\sigma} \cdot \vec{n}_b$ in pure states with (a) $\theta_{ab} = 90^\circ$, (b) $\theta_{ab} = 45^\circ$, (c) $\theta_{ab} = 30^\circ$, (d) $\theta_{ab} = 0^\circ$ respectively. Here, θ_{ab} is the angle between unit vectors \vec{n}_a and \vec{n}_b . The obtained entropic uncertainty relation is optimal: 1. for every point in the shaded area there is a quantum state that gives the corresponding values of $H(A)$ and $H(B)$; 2. for every quantum state, the values of $H(A)$ and $H(B)$ lie in the shadow region.

Taking Eq. (14) into Proposition 1 of Ref. [13], one may easily notice that the Corollary 2 holds, and the equality form of entropic uncertainty relations could also be obtained explicitly from the variance based uncertainty relations for multiple observables [14]. As an illustration, we take Pauli operators of usual qubit system as an example. The variance based uncertainty equality $\Delta\sigma_x^2 + \Delta\sigma_y^2 + \Delta\sigma_z^2 = 4 - 2\text{Tr}[\rho^2]$ (see Refs. [13, 14]) leads to the following entropic uncertainty equality

$$g_\alpha(\sigma_x) + g_\beta(\sigma_y) + g_\gamma(\sigma_z) = 4 - 2\text{Tr}[\rho^2] . \quad (23)$$

Here, the function of entropy g_α is defined in Eq. (14). This gives out an optimal equality form of trade-off relations for $H_\alpha(\sigma_x)$, $H_\beta(\sigma_y)$, $H_\gamma(\sigma_z)$ in arbitrary qubit state, while

results given in Refs.[24, 29, 30] are merely upper and/or lower bounds in the special case of $\alpha = \beta = \gamma = 1$. And for the collision entropy case with $\alpha = \beta = \gamma = 2$, Eq. (23) gives the following uncertainty equality:

$$e^{-H_2(\sigma_x)} + e^{-H_2(\sigma_y)} + e^{-H_2(\sigma_z)} = 1 + \text{Tr}[\rho^2] , \quad (24)$$

where the monotonic relation $\Delta A^2 = g_2(A) = 2 - 2e^{-H_2(A)}$ is employed.

2.3 Uncertainty relations for spin-1 and even higher

Proposition 1 is generally applicable to arbitrary non-degenerate observables, here we take the spin systems as examples for high dimensional systems. For spin-1 operators $J_a = \vec{J} \cdot \vec{n}_a$ with eigenvalues $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$ (assume $\hbar = 1$) we have

$$\ell_2(J_a) = 1 - J_a^2, \quad \ell_3(J_a) = (J_a^2 - J_a)/2. \quad (25)$$

According to Proposition 1, the covariances of the operators $\ell_2(J_a)$ and $\ell_3(J_a)$ can be evaluated and the probability distribution is recovered

$$p_1^2 = \frac{\Omega_{12}\Omega_{13}}{\Omega_{23}} = \frac{[V(J_a^2) + \langle J_a^3 \rangle - \langle J_a^2 \rangle \langle J_a \rangle][V(J_a) - V(J_a^2)]}{4[V(J_a^2) - (\langle J_a^3 \rangle - \langle J_a^2 \rangle \langle J_a \rangle)]}, \quad (26)$$

$$p_2^2 = \frac{\Omega_{12}\Omega_{23}}{\Omega_{13}} = \frac{V(J_a^2)^2 - (\langle J_a^3 \rangle - \langle J_a^2 \rangle \langle J_a \rangle)^2}{V(J_a) - V(J_a^2)}, \quad (27)$$

$$p_3^2 = \frac{\Omega_{13}\Omega_{23}}{\Omega_{12}} = \frac{[V(J_a^2) - (\langle J_a^3 \rangle - \langle J_a^2 \rangle \langle J_a \rangle)][V(J_a) - V(J_a^2)]}{4[V(J_a^2) + \langle J_a^3 \rangle - \langle J_a^2 \rangle \langle J_a \rangle]}. \quad (28)$$

The collision entropy is

$$\begin{aligned} H_2(J_a) &= -\ln[1 - 2(p_1 p_2 + p_1 p_3 + p_2 p_3)] \\ &= -\ln[1 - \frac{1}{2}V(J_a) - \frac{3}{2}V(J_a^2)]. \end{aligned} \quad (29)$$

For two operators J_a and $J_b = \vec{J} \cdot \vec{n}_b$, the entropic uncertainty relation

$$H_2(J_a) + H_2(J_b) \geq c \quad (30)$$

immediately leads to the following variance based uncertainty relation

$$[2 - V(J_a) - 3V(J_a^2)][2 - V(J_b) - 3V(J_b^2)] \leq 4e^{-c} . \quad (31)$$

Puchała, *et al.*[27] had found a simple bound for Eq. (30), i.e.

$$c = -\ln\left[\left(\frac{1+c_{ab}}{2}\right)^4 + \left(1 - \left(\frac{1+c_{ab}}{2}\right)^2\right)^2\right] , \quad (32)$$

where c_{ab} is the maximum overlap of eigenbases of operators J_a and J_b . Considering Eq. (32) for the case of angular momentum operators along the x and z axes, Eq. (31) becomes

$$[2 - V(J_x) - 3V(J_x^2)][2 - V(J_z) - 3V(J_z^2)] \leq \frac{25}{8} - \frac{1}{\sqrt{2}} . \quad (33)$$

A numerical evaluation of above inequality shows that $V(J_x) + V(J_z) \geq 7/16$, which is consistent with that of Ref. [32] for spin-1 system. Similar expression as Eq. (29) may also be obtained for spin- $\frac{3}{2}$ system, of which the entropy reads

$$\begin{aligned} H_2(J_a) = & -\ln \left\{ 1 - \left[\frac{5}{9}V(J_a^3) + \frac{1}{4}V(J_a^2) + \frac{365}{144}V(J_a) \right. \right. \\ & \left. \left. - \frac{41}{18}(\langle J_a^4 \rangle - \langle J_a \rangle \langle J_a^3 \rangle) \right] \right\} . \end{aligned} \quad (34)$$

In principle, there is also no difficult to get similar relations as Eq. (31) for even higher spin systems by applying Proposition 1.

3 Conclusions

We find in this work an equivalence theorem to unify the superficially different classes of uncertainty relations, the variance and entropy based ones. For non-degenerate observables, the probability distributions are recovered from the covariance functions of the operators. Among the various applications of this theorem, optimal entropic uncertainty

relations containing multiple observables are obtained from the variance based uncertainty relations for qubit system, where when the observables are more than two, the obtained entropic uncertainty relations are in equality form. Explicit functional relations between variance and entropy are constructed for higher spin system. While interest in their own right, these results may also have direct applications in the study of quantum nonlocality, as the uncertainty relations are employed to determine the strength of quantum correlations [4, 5]. Another important impact of the equivalence theorem is on the structure of the uncertainty relation in the presence of quantum memory [33], which is crucial for the security of quantum key distribution. Finally, since the theorem generally applies to arbitrary dimensional discrete system, it constitutes the basis for further studies of different uncertainty measures and relations.

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